
Let \( J_n = J_n(2\sqrt{\theta}) \) be the Bessel function of integral order \( n \in \mathbb{Z} \) defined by the generating series

\[
J(z) := e^{\sqrt{\theta}(z-z^{-1})} = \sum_{n \in \mathbb{Z}} J_n(2\sqrt{\theta})z^n.
\]

(a) Show that these functions satisfy the three-term relations

\[
\sqrt{\theta}(J_{n+1} + J_{n-1}) = nJ_n, \quad n \in \mathbb{Z}.
\]

To see this, consider the coefficient by \( z^n \) in \( \frac{\partial}{\partial z} J(z) \) (this is a standard technique of working with generating functions).

(b) Define the kernel

\[
K(x,y) := \sum_{s=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...} J_{x+s}J_{y+s}.
\]

Show (by multiplying the sum by \( x - y \) and using the three-term relation) that this sum is equal to

\[
K(x,y) = \sqrt{\theta} \frac{J_{x-\frac{1}{2}}J_{y+\frac{1}{2}} - J_{x+\frac{1}{2}}J_{y-\frac{1}{2}}}{x - y}.
\]

Another way: show that \( K(x+1,y+1) - K(x,y) \) is the same for both (1) and (2).

(c) Using the contour integral representation of a Laurent coefficient

\[
J_n = \frac{1}{2\pi i} \oint_{|z|=1} J(z) \frac{dz}{z^{n+1}}, \quad n \in \mathbb{Z},
\]

derive the following double contour integral expression for the kernel:

\[
K(x,y) = \frac{1}{(2\pi i)^2} \int \int_{|z|>|w|} J(z) J(1/w) \frac{dzdw}{z - w} \frac{dzdw}{z^{x+\frac{1}{2}}w^{y+\frac{1}{2}}}.
\]

This is done by using the summation representation for \( K(x,y) \) and by interchanging the contour integration and the summation.

Another way to solve the problem is to fill in the details in the argument of \[ \text{http://arxiv.org/abs/math/0309074}, \text{page 19}. \]
This exercise is to show that behavior of the determinantal process with the discrete sine kernel differs from that of the Bernoulli process of independent particles.

(a) Consider the discrete sine process $S$ on $\mathbb{Z}$ which appears at the global position $u = 0$ in the bulk of the Plancherel random partition. Its kernel is

$$
K(x, y) = \begin{cases} 
1/2, & y = x, \\
0, & y - x \text{ is even } \neq 0, \\
\frac{(-1)^{(y-x)-1/2}}{\pi(y-x)}, & y - x \text{ is odd.}
\end{cases}
$$

What is the density of particles under this sine process $S$?

Let $B$ be the Bernoulli process on $\mathbb{Z}$ with the same density of particles. That is, every position of $\mathbb{Z}$ is occupied by a particle independently of others with the same probability $p$ which is the density.

(b) Compare (numerically) the probabilities to see three consecutive particles \( \bullet \bullet \bullet \) under $S$ and $B$. Which is bigger?

(c)* Take some software and compute the probabilities of events \( \bullet \ldots \bullet \) for $S$ for several more values $n = 4, 5, 6 \ldots$. Compare this with the same probabilities for the Bernoulli process $B$.

(d) Now take the event \( \bullet \circ \bullet \circ \). Explain how to compute its probability under $S$ with the help of the determinantal structure and the complementation principle (Lecture Notes, p.92). Compute (numerically) this probability under $S$ with the same probability under $B$.

(e) Do the same as in (d) for the event \( \circ \circ \bullet \bullet \).

(f)* Take some software and compute the probabilities under $S$ of the events \( \bullet \circ \bullet \circ \ldots \bullet \circ \) and \( \circ \circ \circ \ldots \bullet \bullet \) for several more values $n = 3, 4, 5, \ldots$. Compare them with the same probabilities for the Bernoulli process $B$.

As a result of the exercise, you should see that under the sine process $S$ the particles tend to be more equally spaced than under the Bernoulli process $B$. 