Random tiling = beautiful subject, uses combinatorics, probability, analysis,... has physical sense, too.

Today: 2 examples of problems & enumeration.

Domino tiling by 1x2 dominos - of a region of the plane

Rect.

Dimer coverings of a graph

Fibonacci

physical sense: let us think of lany molecules which pair with 1 another & they are on a lattice

\( a \times N \) even these pairings have some energy (energy = log # config.)

\( a \) want to study large systems

Aztec diamond

(same question)

Random tiling of aztec (1990s)

arctic circle

(Show on comp)

q: how many?

Originally: EKUP, Jun 1991 arXiv by biject. w. some RT/algebraic objects & introduced Aztec
And there are many generaliz
of the # by weightign
edges of the graph.

\[ \text{\texttt{H}} \quad 2 \quad \frac{n(n+1)}{2} \]
\[ \text{(EKLIP)} \]
\[ n=3 \]

\[ a \quad b \quad c \quad d \quad e \quad f \]
\[ \Rightarrow acg + bde + efg \]
\[ = \text{partition of} \ f. \]

Another bijection: bij of Aztec diamond filling
to square ice configurations

\[ \text{Square ice} \]

(2015: Lee
between 2
graphene
layers)

\[ \Rightarrow \]

at each vertex:
2 H's close and 2
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\[ \Rightarrow \]
Proof of aztec diamond theorem.

Lemma (Urban renewal) J. Propp, 1990s / Cluster alg. FÁ 2005

\[ \sum \text{Weights of matchings} = \sum \text{Weights of } \]

\[ \begin{align*}
\alpha & = \frac{A}{R} \\
\beta & = \frac{B}{R} \\
\gamma & = \frac{C}{R} \\
\delta & = \frac{D}{R}
\end{align*} \]

Proof:

\[ I = \frac{ad+bc}{R} \]

(i.e. small) \times R = (\varepsilon \text{ large})
Proof of the aztec diamond theorem

(Initial graph)

\( n^2 \) "cells"

\( \Rightarrow \) \( 4n^2 \) edges total

\( 2n \) edges non-triangular

\[ \text{split each vertex into 3: does not change weight} \]

\( \text{apply L, 4 times} \)

\[ x = \frac{1}{2} \]

\( \frac{1}{2} \times 2^4 = \left( \frac{1}{2^n} \right) \)

\[ \text{graph of ord 1, every tiling has 2 dominoes} \]

\[ \Rightarrow \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \times 2 \]

\[ 2 \]

\[ \frac{1}{2} n(n-1) \]
Generalize: $P($edge is in dom-tile$)$

\[ \downarrow \]

just put $X$

\[ (\text{var.}) \]

on one of the

edges & look

at urban renewal

process.

(sum principle--)

\(\text{\ding{182}}\) For rect., domino tilings

\[ \Rightarrow 3 \times 3 \ \text{rect.} \]
Couple of words on lozenge tilings

Aztec - 1991
Hexagon - 1980s!

But more recently connections to RT
were revisited & understood

Tilings enumerate basis in repr. of \( U(a+c) \)
(irrep. corr. to \( \lambda = \left( \frac{b, b, \ldots, b, 0, 0, \ldots, 0}{a, c} \right) \))

i.e. basis in a vector space w. action of
unitary group by
(\textit{unitary}) transformation
w. no inv. subspaces.
(prop)

\( S_{\chi}(x_1, \ldots, x_N) = \frac{\det [x_i^{a_j+a_j-\delta_j}]}{\det [x_i^{N-j}]} \)
\( \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N \)

Example:

Branching:

\( S_\chi(x_1, \ldots, x_{N-1}) \)
\( = \sum_{\chi} S_\chi(x_1, x_{N-1}) \)