Known generating functions.

\( (1 + x)^\alpha = 1 + \frac{\alpha^1}{1!} x + \frac{\alpha^2}{2!} x^2 + \frac{\alpha^3}{3!} x^3 + \ldots, \)

where \( \alpha \in \mathbb{C} \) and \( \alpha^k := \alpha(\alpha - 1) \ldots (\alpha - k + 1), \ k = 1, 2, \ldots \ (k \text{ factors}); \) we set \( \alpha^0 := 1. \)

b) \( e^x = \exp x = 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots; \)

c) \( \log \left( \frac{1}{1-x} \right) = x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \ldots; \)

d) \( \sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \ldots; \)

e) \( \cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \ldots. \)

Operations with generating functions/power series. Let \( A(x) \) and \( B(x) \) be the generating functions of the sequences \( \{a_k\}_{k=0}^{\infty} \) and \( \{b_k\}_{k=0}^{\infty}, \) respectively.

- (summation) The sum of the generating functions \( A(x) \) and \( B(x) \) is, by definition,
  \[ A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots; \]

- (multiplication) The product of the generating functions \( A(x) \) and \( B(x) \) is, by definition,
  \[ A(x)B(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \ldots; \]

- (substitution) If \( B(0) = b_0 = 0, \) it is possible to define the generating function \( A(B(x)) \) as
  \[ A(B(x)) := a_0 + a_1b_1x + (a_1b_2 + a_2b_1^2)x^2 + (a_1b_3 + 2a_2b_1b_2 + a_3b_1^3)x^3 + \ldots. \]

- (formal differentiation and integration) The formal derivative of a generating function \( A(x) \) is defined as
  \[ A'(x) := a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots. \]

The formal anti-derivative (primitive, indefinite integral) is defined as
\[ \int A(x) := a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} + a_3\frac{x^4}{4} + \ldots. \]

For polynomials all these operations are just the usual ones (but be careful about the right constant in the integration!).

If \( A(x) \) is a power series as above, we denote \([x^n]A(x) := a_n, \) for any nonzero \( \alpha \) we define \([\alpha x^n]A(x) := \frac{1}{\alpha}a_n. \)

**Seminar problems**

**Problem 2.1** (Generating functions for basic combinatorial quantities). Let \( n \) be a fixed integer. Find the generating functions for the sequences:

a) \( 1, n, n^2, n^3, \ldots \) (\( a_k = n^k, \ k = 0, 1, \ldots; \))

b) \( \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}, 0, 0, 0, \ldots \) (\( a_k = \binom{n}{k}, \ k = 0, 1, \ldots; \))

c) \( \binom{n-1}{0}, \binom{n}{1}, \binom{n+1}{2}, \binom{n+2}{3}, \binom{n+3}{4}, \ldots, \) (\( a_k = \binom{n+k-1}{k}, \ k = 0, 1, \ldots; \))

**Problem 2.2**. Find the generating functions of the following sequences \( \{a_k\}_{k=0}^{\infty} \): 

a) \( 1, 0, 0, 0, \ldots; \)

b) \( a_k = 1, \ k = 0, 1, 2, \ldots; \)
Problem 2.17. Show that for a generating function we have
\[
\text{Problem 2.16. A}\] Show that the functions
\[
\text{Problem 2.15. Let the generating function}\]
\[
\text{Problem 2.14. Let } f(x) \text{ be a generating function } f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \text{ with } f(0) \neq 0. \text{ Show that there exists a generating function } g(x) \text{ such that } f(x)g(x) = 1.
\]
\[
\text{Problem 2.13. What problems one encounters trying to define the generating function } A(B(x)) \text{ if } B(0) = b_0 \neq 0? \text{ Why this cannot be done in a purely algebraic way?}
\]
\[
\text{Problem 2.12. Find the coefficient of } x^n \text{ in the power series expansion of } f(x) = \frac{1}{(1-x)^2} \text{ by writing } f(x) \text{ as a sum of partial fractions.}
\]
\[
\text{Problem 2.11. Find first three terms of the generating function } f^{-1}(x), \text{ where } f(x) \text{ is given by}
\]
\[
\text{Problem 2.10. Calculate the first three coefficients of the generating function } \frac{1}{\cos x}.
\]
\[
\text{Problem 2.9. Recall our recurrence for the binomial coefficients: } \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \text{ Use it to find the following generating function in two variables: } F(x,y) := \sum_{n,k} \binom{n}{k} x^k y^n.
\]
\[
\text{Problem 2.8. For each of the sequences in Problem 2.7 find the exponential generating function.}
\]
\[
\text{Problem 2.7. Solve the following recurrence relations (first, find the generating functions for the sequences, and then from them deduce formulas for the } n^{th} \text{ term of the sequence):}
\]
\[
\text{Problem 2.6. Find a) } [x^n] e^{2x}; \quad \text{b) } [x^n/n!] e^{3x}; \quad \text{c) } [x^n] \frac{1}{(1-ax)(1-bx)} \quad \text{(here } a \neq b).}
\]
\[
\text{Problem 2.5. Let } f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \text{ be the generating function of the sequence } \{a_k\}_{k=0}^\infty. \text{ Express simply, in terms of } f(x), \text{ the generating functions of the following sequences (here } k = 0, 1, 2, \ldots, )
\]
\[
\text{Problem 2.4. Prove (using only formal operations with known generating functions and some known combinatorial identities) that } e^x e^{-x} = 1.
\]
\[
\text{Problem 2.3. Let } P(z) := b_0 + b_1 z + \ldots + b_m z^m \text{ be a given polynomial of degree } m. \text{ Find the generating function of the sequence } \{P(k)\}_{k=0}^\infty.
\]
\[
c) a_k = k + 1, k = 0, 1, 2, \ldots;
\]
\[
d) 1 \cdot 2, 2 \cdot 3, 3 \cdot 4, \ldots;
\]
\[
e) a_k = 7^k, k = 0, 1, 2, \ldots;
\]
\[
f) \frac{1}{12}, \frac{1}{23}, \frac{1}{34}, \frac{1}{45}, \ldots.
\]
**Problem 2.18.** Find the generating functions of the following sequences $\{a_k\}_{k=0}^{\infty}$:

- a) $(1)$ $-16, 0, 0, 0, 1, 0, 0, \ldots$;
- b) $(1)$ $a_k = (-1)^k$, $k = 0, 1, 2, \ldots$;
- c) $(1)$ $a_k = \alpha k + \beta$, $k = 0, 1, 2, \ldots$ (here $\alpha, \beta \in \mathbb{C}$ are constants);
- d) $(1)$ $a_k = (k + 1)^2$, $k = 0, 1, 2, \ldots$;
- e) $(2)$ $a_k = \alpha k^2 + \beta k + \gamma$, $k = 0, 1, 2, \ldots$ (here $\alpha, \beta, \gamma \in \mathbb{C}$ are constants);
- f) $(1)$ $a_k = 8 \cdot 7^k - 3 \cdot 4^k$, $k = 0, 1, 2, \ldots$.

**Problem 2.19** (0.5 for each sequence; 6 total). For all sequences in Problems 2.2 and 2.18 find their exponential generating functions.

**Problem 2.20** (2). Let $P(z)$ be a given polynomial of degree $m$. Find the exponential generating function of the sequence $\{P(k)\}_{k=0}^{\infty}$.

**Problem 2.21**. Prove (using only formal operations with known generating functions and some known combinatorial identities):

- a) $(1)$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$; $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ (here $i^2 = -1$);
- b) $(2)$ $(\sin x)^2 + (\cos x)^2 = 1$.

**Problem 2.22.** Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ be the generating function of the sequence $\{a_k\}_{k=0}^{\infty}$. Express simply, in terms of $f(x)$, the generating functions of the following sequences (here $k = 0, 1, 2, \ldots$)

- a) $(1)$ $\{\alpha a_k + \beta\}$ ($\alpha$ and $\beta$ are constants);
- b) $(2)$ $\{(\alpha k^2 + \beta k + \gamma)a_k\}$;
- c) $(1)$ $0, 0, 1, a_3, a_4, a_5, \ldots$;
- d) $(1)$ $a_0, 0, a_1, 0, a_2, 0, a_3, 0, a_4, 0, \ldots$;
- e) $(2)$ $\{a_{k+h}\}$ (where $h$ is a given positive integer);
- f) $(1)$ $\{a_{k+2} - a_{k+1} - a_k\}$.

**Problem 2.23** (0.5 for each sequence; 6 total). Let $f(x) = \frac{a_0}{0!} + \frac{a_1}{1!} x + \frac{a_2}{2!} x^2 + \ldots$ be the exponential generating function of the sequence $\{a_k\}_{k=0}^{\infty}$. Express simply, in terms of $f(x)$, the generating functions of all the sequences from Problems 2.5 and 2.22.

**Problem 2.24.** (a) (2) Let $P(z)$ be a given polynomial of degree $m$ and let $f(x)$ be the generating function of the sequence $\{a_k\}_{k=0}^{\infty}$, where $f(x)$ is given by $\frac{P(z)}{z^m}$. Express in terms of $f(x)$ the generating function of the sequence $\{P(k) a_k\}_{k=0}^{\infty}$.

- (b) (2) Solve the same problem for $f$ being the exponential generating function: express in terms of $f(x)$ the exponential generating function of the sequence $\{P(k) a_k\}_{k=0}^{\infty}$.

**Problem 2.25.** Find

- a) $(1)$ $[x^n/n!] e^{ax}$;
- b) $(1)$ $[x^n/n!] \sin x$;
- c) $(2)$ $[x^n](1 + x^2)^m$.

**Problem 2.26.** Solve the following recurrence relations (first, find the generating functions for the sequences, and then from them deduce formulas for the $n$th term of the sequence):

- a) $(1)$ $a_{n+1} = 3a_n + 2$, $n \geq 0$, $a_0 = 0$;
- b) $(1)$ $a_{n+1} = \alpha a_n + \beta$, $n \geq 0$, $a_0 = 0$;
- c) $(1)$ $a_{n+2} = 2a_{n+1} - a_n$, $n \geq 0$, $a_0 = 0$, $a_1 = 1$.

**Problem 2.27** (2). Derive the formula for the generating function $F(x, y)$ from Problem 2.9 using the known one-variable generating function for the binomial coefficients, see Problem 2.1(a).

**Problem 2.28.** Calculate the first three coefficients of the generating function $\frac{1}{f(x)}$, where $f(x)$ is given by

- a) $(1)$ $f(x) = (1 + x)^m$;
Problem 2.29. Find first three terms of the generating function \( f^{-1}(x) \), where \( f(x) \) is given by

a) \((1) f(x) = \tan x;\)
b) \((1) f(x) = x + 2\sqrt{1 + x};\)
c) \((2) f(x) = \log(1 - x).\)

Problem 2.30. Let \(\alpha_{n} \) be the number of subsets of \(\{1, \ldots, n\}\) that contain no two consecutive elements \((n = 1, 2, \ldots)\). Find the recurrence that is satisfied by these numbers, and then find the numbers \(\alpha_{n}\).

Problem 2.31. Solve Problem 2.12 in another way (e.g., by using the division of polynomials or some other trick).

Problem 2.32. (a) \((1)\) Let \(A(x)\) and \(B(x)\) be the generating functions of the sequences \(\{a_{k}\}_{k=0}^{\infty}\) and \(\{b_{k}\}_{k=0}^{\infty}\), respectively. Write down the sequence for which the generating function is \(A(x)B(x)\).

(b) \((1)\) Let \(A(x)\) and \(B(x)\) be the exponential generating functions of the sequences \(\{a_{k}\}_{k=0}^{\infty}\) and \(\{b_{k}\}_{k=0}^{\infty}\), respectively. Write down the sequence for which the exponential generating function is \(A(x)B(x)\).

Problem 2.33. \((1)\). Show that there does not exist a generating function \(f(x)\) such that \(xf(x) = 1\).

Problem 2.34. \((1)\). Show that if two generating functions \(A(x)\) and \(B(x)\) are nonzero (the zero generating function has all coefficients equal to zero), then their product \(A(x)B(x)\) is also nonzero.

Problem 2.35. \((4)\). Show that in Problem 2.15 the functions \(A(x)\) and \(C(x)\) coincide.

We denote \(A(x) = C(x)\) by \(B^{-1}(x)\) and call this the inverse of \(B(x)\).

Problem 2.36. \((2)\). Prove the formula \((f(x)g(x))' = f'(x)g(x) + f(x)g'(x)\).

Problem 2.37. \((2)\). Show that \(\int (f'(x)g(x) + f(x)g'(x)) = f(x)g(x) - f(0)g(0)\).

 Supplementary problems

Problem 2.38. Prove the formal power series identities:

a) \((2)\) \((1 + x)^{\alpha}(1 + x)^{\beta} = (1 + x)^{\alpha + \beta};\)
b) \((2)\) \(\exp(\log((1 - x)^{-1})) = (1 - x)^{-1};\)
c) \((2)\) \(\log(1 + x) = x - \frac{1}{2}x^{2} + \frac{1}{3}x^{3} - \cdots + \frac{(-1)^{n+1}}{n}x^{n} + \cdots;\)
d) \((2)\) \(\log((1 - x)^{\alpha}) = \alpha \log(1 - x)\).

Problem 2.39. \((1\) for each sequence; \(3\) total). For each of the sequences in Problem 2.26 find the exponential generating function (you may need to solve a differential equation to do so!).

Problem 2.40. \((3)\). Let \(f(x)\) be a generating function such that \(f''(x) + f(x) = 0\). Give a careful proof of the fact that \(f(x) = C_{1}\sin x + C_{2}\cos x\)

Problem 2.41. \((1\) for each sequence, 7 total). Find the generating functions of the following sequences:

a) \((n+1)^{\infty}_{n=0};\)
b) \((1)^{\infty}_{n=1}\); c) \(1, 0, 1, 0, 1, 0, \ldots\); d) \((n+1)!^{\infty}_{n=2};\)
e) \((1/(n+1))^{\infty}_{n=0};\)
f) \(F_{1}, 2F_{2}, 3F_{3}, 4F_{4}, \ldots\);
where \(F_{n}\) are the Fibonacci numbers;
g) \((n^{2} + n + 1)/n!^{\infty}_{n=1}\).

Problem 2.42. \((4)\). For given integers \(n, k\) let \(b_{n,k}\) be the number of \(k\)-subsets (i.e., subsets with \(k\) elements) of \(\{1, \ldots, n\}\) that contain no two consecutive elements. Find a recurrence for \(b_{n,k}\), find a suitable generating function and then find the numbers themselves.

Problem 2.43. \((2)\). By comparing the results of Problems 2.30 and 2.42 deduce an identity.

Problem 2.44. \((2)\). Show that the generating functions

\[a_{1}x + a_{2}x^{2} + a_{3}x^{3} + \ldots, \quad a_{1} \neq 0,\]
form a group with respect to the composition operation \((A_{1} \circ A_{2})(x) := A_{1}(A_{2}(x))\).