COMBINATORICS. PROBLEM SET 8. DIRICHLET GF’S

SEMINARY PROBLEMS

Problem 8.1. Find a quick way of writing out all prime numbers from 2 to a given number \( N \).

\[ \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \] is the Riemann zeta function.

Problem 8.2. Show that \( \zeta(s) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \).

Let \( f : \mathbb{N} \to \mathbb{C} \) be a function of a natural number (such functions are called arithmetic). (Equivalently, we can speak of the sequence \( \{f(1), f(2), f(3), \ldots\} \).) Associate to \( f \) the Dirichlet GF: \( A(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \).

Problem 8.3. Let \( A(s) \) and \( B(s) \) be the Dirichlet GF’s for two arithmetic functions \( f \) and \( g \), respectively. Find (in terms of \( f \) and \( g \)) the sequence whose Dirichlet GF is \( A(s)B(s) \).

Problem 8.4. Let for \( n \in \mathbb{N} \) by \( d(n) \) denote the number of divisors of \( n \). E.g., \( d(6) = 4 \). Find the Dirichlet GF for \( d(n) \).

An arithmetic function \( f \) is called multiplicative iff \( f(ab) = f(a)f(b) \) for any relatively prime \( a, b \).

Problem 8.5. Show that a multiplicative arithmetic function is completely defined by its values on powers of prime numbers.

Problem 8.6. Show that an arithmetic function \( f \) is multiplicative iff the corresponding Dirichlet GF can be represented in the following form: \( \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_{p \text{ prime}} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \ldots) \). (In fact, this identity for multiplicative arithmetic functions is useful in many of the homework problems.)

Problem 8.7. Let \( \mu(n) \) be the arithmetic function with Dirichlet GF which is the inverse of the Riemann zeta function: \( \frac{1}{\zeta(s)} = \prod_{p \text{ prime}} (1 - p^{-s}) \). Find a \( \mu(n) \) for all \( n \) in an explicit form.

Problem 8.8. Use the inclusion-exclusion principle to show that \( \left( \sum_{n=1}^{\infty} \frac{1}{n^s} \right) \left( \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right) = 1 \).

Problem 8.9. Use Dirichlet GF’s to prove the Möbius inversion theorem:

\[ f(n) = \sum_{d \mid n} g(d) \quad \Leftrightarrow \quad g(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right)f(d). \]

for any arithmetic functions \( f \) and \( g \). (In fact, this theorem is useful in many of the homework problems.)

Problem 8.10. Prove the inclusion-exclusion principle: \( B \) is a finite set, each element can possess or not possess some of the properties \( c_1, \ldots, c_m \). Let \( N(c_1, \ldots, c_k) \) be the number of elements each of which possesses the properties \( c_1, \ldots, c_k \). Let also \( N(1) = \#B \). Then the number of elements in \( B \) which do not possess any of the properties \( c_1, \ldots, c_m \), is

\[ N(1) - N(c_1) - \cdots - N(c_m) + N(c_1, c_2) + \cdots + N(c_{m-1}, c_m) - N(c_1, c_2, c_3) - \ldots \]

Problem 8.11. Show that the Dirichlet GF’s of nonzero multiplicative arithmetic functions form a group with respect to the multiplication of Dirichlet GF’s.
Problem 8.12. A ticket has a 6-digit number $abcdef$. Ticket is lucky iff $a + b + c = d + e + f$. Find (using the inclusion-exclusion principle) the total number of lucky tickets. (Hint 1: by using the bijection $abcdef \to abc(9 - d)(9 - e)(9 - f)$, conclude that the number of lucky tickets is the same as the number of tickets with sum of the digits equal to 27. Hint 2: assume that $a, b, c, d, e, f \geq 0$ and use the inclusion-exclusion principle with $c_i$ being the property that the $i$th number in a ticket is $\geq 10$.)

Problem 8.13. A disorder on the set $\{1, \ldots, n\}$ is a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $\sigma(k) \neq k$ for any $k$. Find (using the inclusion-exclusion principle) the number of disorders on $\{1, \ldots, n\}$. (Hint: $c_i$ is the property of a permutation to fix $i$, i.e., $\sigma(i) = i$.)

Homework

Problem 8.14. Use the inclusion-exclusion principle to show that 
$$\max(a_1, \ldots, a_n) = a_1 + \cdots + a_n - \min(a_1, a_2) - \cdots - \min(a_{n-1}, a_n) + \min(a_1, a_2, a_3) + \cdots + (-1)^n \min(a_1, \ldots, a_n).$$
(Specify the properties $c_i$ explicitly.)

Problem 8.15. Let $\varphi(n)$ be the number of numbers among $\{1, \ldots, n-1\}$ which are relatively prime to $n$. Show that for $n = p_1^{k_1} \cdots p_m^{k_m}$ ($p_i$ distinct primes) we have $\varphi(n) = n(1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_m})$. Use the inclusion-exclusion principle. (Specify the properties $c_i$ explicitly.)

Problem 8.16. From the above problem, show that the arithmetic function $\varphi$ is multiplicative.

Problem 8.17. Find the Dirichlet GF for $\varphi$.

Problem 8.18. How many of the following $n^2$ fractions
$$\begin{array}{cccc}
1/1 & 1/2 & 1/3 & \ldots & 1/n \\
2/1 & 2/2 & 2/3 & \ldots & 2/n \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
n/1 & n/2 & n/3 & \ldots & n/n
\end{array}$$
are irreducible? Use the inclusion-exclusion principle. (Specify the properties $c_i$ explicitly.)

Problem 8.19. Let $\sigma(n)$ be the sum of all the divisors of $n$ (n included), e.g., $\sigma(6) = 12$. Show that this is a multiplicative arithmetic function.

Problem 8.20. Find the Dirichlet GF for the arithmetic function $\sigma$.

Problem 8.21. Show that $\sum_{\delta: \delta \mid n} \mu(\delta)d(\frac{n}{\delta}) = 1$ for $n \geq 1$. (Here $\mu$ and $d$ are defined above.)