COMBINATORICS. MIDTERM TEST

1. BASICS AND GF’S

Problem 1. Find \([x^{3n}]e^{2x}\) (the coefficient by \(x^{3n}\) in the power series expansion of \(e^{2x}\)).

Problem 2. Show that

\[
\binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + \ldots + 2^k\binom{n}{k} + \cdots + 2^n\binom{n}{n} = 3^n.
\]

Problem 3. What is the total number of ways in which the letters in the word “PETERSBURG” can be rearranged?

Problem 4. Solve the recurrence (using generating functions):

\[a_{n+2} = 3a_{n+1} + 4a_n - 6n - 1 \quad (n \geq 0), \quad a_1 = 4, \quad a_0 = 2.\]

(Advice: substitute your answer into the equation to check yourself.)

Problem 5. Compute the sum:

\[
\sum_{n=1}^{\infty} \frac{10^n + 3n}{n!}
\]

(Advice: note the range of summation.)

2. CATALAN NUMBERS

Of these two problems, solve one (of your choice)

Problem 6. Show that the number of Young diagrams which are inside the staircase shaped Young diagram \((n-1, n-2, \ldots, 1)\) is \(\text{Catalan}_n\). (Hint: bijection with certain up-right paths.)

Problem 7. Show that the number of tiling of the staircase shape \((n, n-1, n-2, \ldots, 2, 1)\) with \(n\) rectangles is \(\text{Catalan}_n\). (Hint: derive the Catalan recurrence relation by deleting one of the tiling rectangles.)

3. LAGRANGE INVERSION

Problem 8. Write a solution to the cubic equation \(x - 1 = ax^3\), that is, find the generating series \(x = x(a)\).

4. ASYMPTOTICS

Problem 9. Let \(a_n\) be a sequence with the generating function

\[f(x) := \sum_{n=0}^{\infty} a_n x^n = \frac{2}{(1-x)(1-2x)(1-3x)}.\]

(a) What is the radius of convergence of the series \(\sum_{n=0}^{\infty} a_n x^n\)? (Hint: use singular points.)

(b) Find minimal possible \(R\) such that \(a_n\) grows slower than \((\frac{1}{R} + \epsilon)^n\) for any \(\epsilon > 0\).

(c) Expand \(f(x)\) as a sum of partial fractions. Find an explicit formula for \(a_n\).

Problem 10. Let \(a_n = \binom{4n}{2n}\).

(a) Make sure that the sequence \(a_n\) is hypergeometric. Deduce hypergeometric-type asymptotics for the sequence \(a_n\).

(b) Using Stirling’s formula, find exact asymptotics of the sequence \(a_n\).
5. Supplementary Problems

Problem 11. Give a combinatorial proof of (1) in Problem 2 (Hint: interpret $3^n$ as the number of all sequences of length $n$ with letters $a, b, c$).

Problem 12. Solve the differential equation using generating functions:

$$f'(x) + xf(x) = 0.$$  

Advice: substitute your answer into the equation to check yourself.

Problem 13. Show that any minor (= determinant formed by some of the rows $i_1, \ldots, i_k$ and the same number of columns $j_1, \ldots, j_k$, $k$ is arbitrary) of the matrix

$$\begin{bmatrix}
1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} & \alpha^N \\
& 0 & 1 & \alpha & \cdots & \alpha^{N-2} & \alpha^{N-1} \\
& & \cdots & \cdots & \cdots & \cdots & \cdots \\
& 0 & 0 & 0 & \cdots & 1 & \alpha \\
& 0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}$$

is nonnegative (here $\alpha > 0$). (Hint: use nonintersecting paths approach.)